L16 Feb12 Quotient

Thursday, February 12, 2015 11:07 AM

Quotient Topology

Given $(X, \mathcal{T}_X)$, either $\sim$ or $q : X \xrightarrow{\text{onto}} Q$
$$Q = X/\sim$$

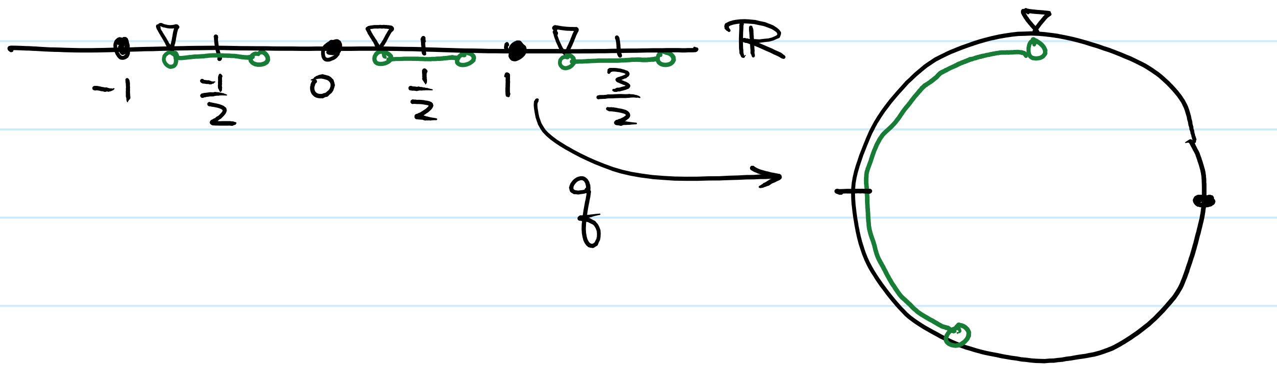
The quotient topology $\mathcal{T}_q$ on $X/\sim$ or Q

$$\mathcal{T}_q = \{ V \subset X/\sim \, : \, q^{-1}(V) \in \mathcal{T}_X \}$$

Circles

1. circle as $[0,1]/\sim$
2. $X = \mathbb{R}$, $\mathcal{T}_X = \mathcal{T}_{std}$

   $x \sim y$ if $x - y \in \mathbb{Z}$



3. All the above are the "circle"

$$[0,1]/\sim \;\xleftrightarrow[\text{homeo.}]{}\; \mathbb{R}/\mathbb{Z} \;\xleftrightarrow[\text{homeo}]{}\; S^1$$

$$S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$$

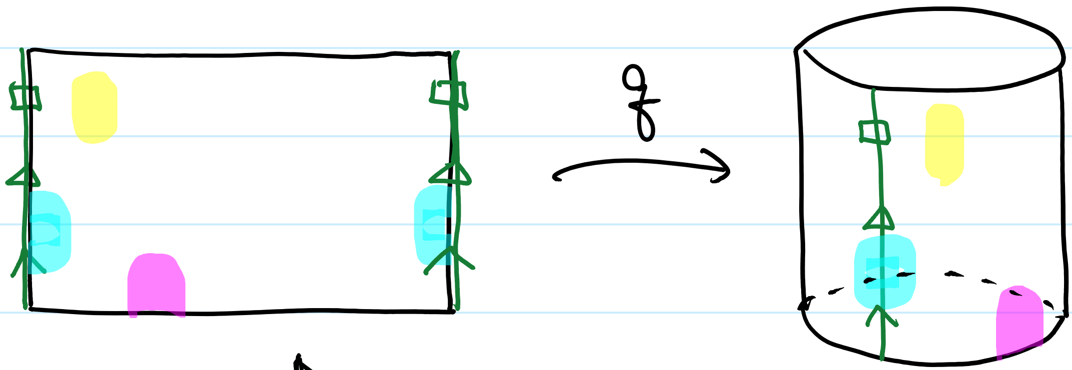
$$[x] \longmapsto e^{2\pi i x}$$

4. Similarly, we have **cylinder**

$([0,1]\times[0,1])/\sim$ where

$(s_1, s_2) \sim (t_1, t_2)$ if $\begin{cases} |s_1 - t_1| = 0, 1 \\ s_2 = t_2 \end{cases}$

Gluing only on the 1st coordinate



homeo

$$\left.\frac{\mathbb{R}\times[0,1]}{\mathbb{Z}\times 0}\right. \xleftrightarrow{\text{homeo}} S^1\times[0,1] \subset \mathbb{R}^3$$

5. **Torus**

Recall that it can be seen as

* surface of revolution $\subset \mathbb{R}^3$
* $S^1 \times S^1$, product topology

$([0,1]\times[0,1])/\sim$ where

$(s_1, s_2) \sim (t_1, t_2)$ if

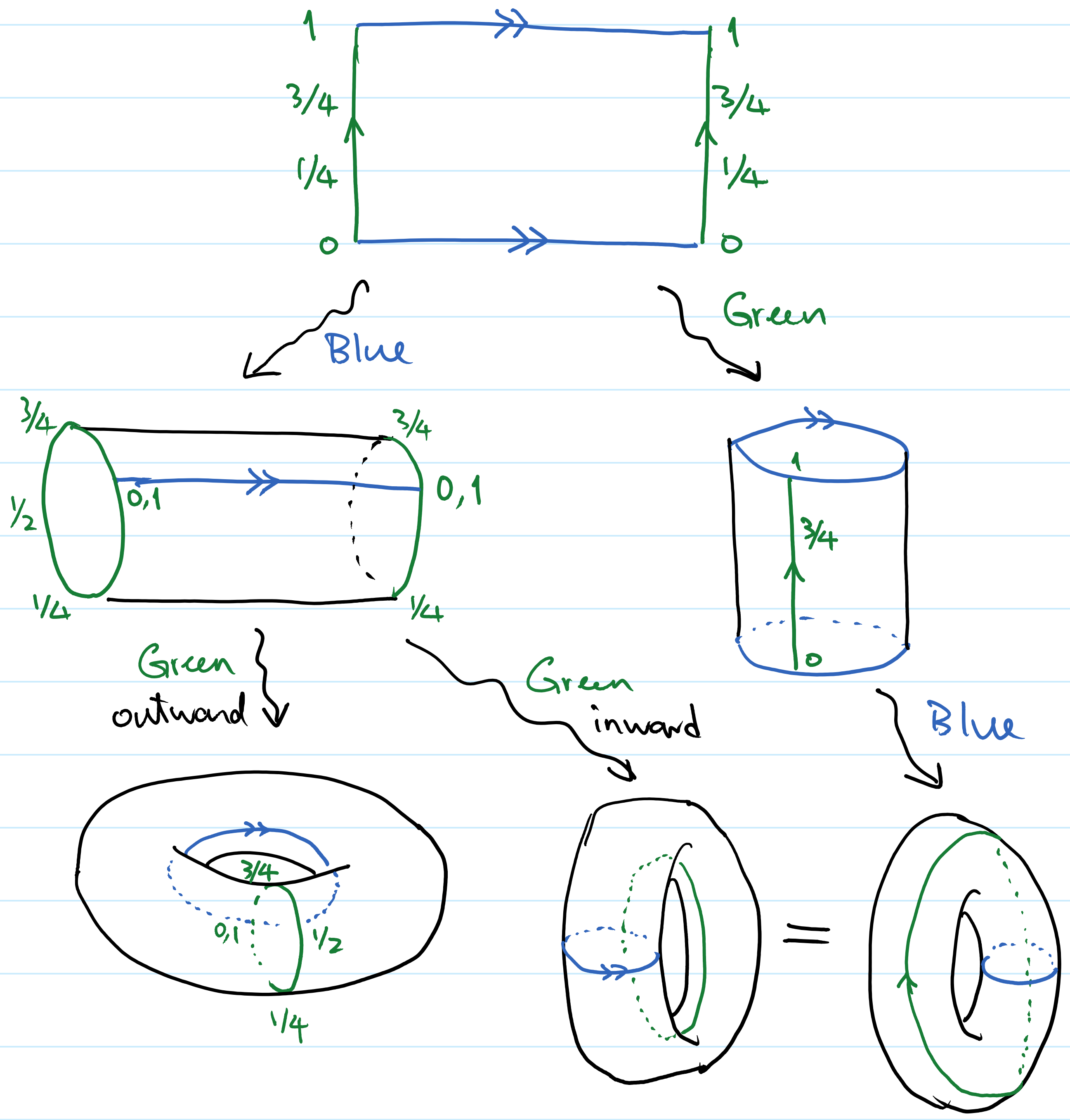
$\begin{cases} |s_1 - t_1| = 0, 1 , \\ |s_2 - t_2| = 0, 1 \end{cases}$

$\updownarrow$

$\mathbb{R}^2/\mathbb{Z}^2$

L16 p3

Monday, February 16, 2015 8:20 PM



Note that from above pictures, apparently the result depends on how it is glued. In fact, all are homeomorphic, just "placed" differently in $\mathbb{R}^3$.

6. $\mathbb{R}^n / \mathbb{Z}^n = S^1 \times \cdots \times S^1$, n-dim Torus

7. Mobius strip or band

$[0,1]\times[0,1]/\sim$ where $(s_1, s_2) \sim (t_1, t_2)$ if

$(s_1, s_2) = (t_1, t_2)$ or $\begin{cases} |s_1 - t_1| = 0, 1 \\ s_2 = 1 - t_2 \end{cases}$

For simplicity, often say

identify $(0, t)$ with $(1, 1-t)$ on $[0,1]^2$

8. Klein Bottle

Identify $(s, 0)$ with $(s, 1)$ and

$(0, t)$ with $(1, 1-t)$ on $[0,1]^2$

Note. Klein Bottle $\not\subset \mathbb{R}^3$

But basic neighborhoods of any point

$\| \text{homeo.}$

$\{z \in \mathbb{C} : |z| < 1\}$

9. Projective Plane, $\mathbb{RP}^2$

Identify $(s, 0)$ with $(1-s, 1)$ and

$(0, t)$ with $(1, 1-t)$ on $[0,1]^2$

OR

Identify z with $-z$ if $|z| = 1$ on

$\{z \in \mathbb{C} : |z| \le 1\}$

Exercise. Show they are homeomorphic.